

# Acceleration of Fractional Fourier Transforms via Tensor-train Decomposition

Runjia Zhang<sup>1</sup>, Chao Li<sup>1</sup>

<sup>1</sup>Tensor Learning Team, RIKEN Center for Advanced Intelligence Project, Tokyo, Japan  
 {runjia.zhang, chao.li}@riken.jp,

## Abstract

Fractional Fourier transform (FrFT) is a generalization of the ordinary Fourier transform. In this extended abstract, we discuss tensor network enabled method to accelerate the numerical calculation of discrete FrFT, along other numerical and optical realizations. Also we discuss how the use of the proposed FrFT approach extends to optics, signal processing and differential equations, such as the Schrödinger equation.

## 1 Introduction

Fractional Fourier transform (FrFT) is the fractional power of the Fourier transform with an order parameter  $a$ . FrFT is of utility in every area that ordinary Fourier transform has shown its capability, such as signal and image processing and time-varying differential equations [7; 9; 13]. It is also solution to important differential equations, such as the Schrödinger equation. We now define the  $a$ th order FrFT  $f_a(u)$  through the following linear integral transform:

$$f_a(u) = \int K_a(u, v) f(v) dv, \quad (1)$$

where

$$K_a(u, v) = A_\phi \exp[i\pi(\cot\phi u^2) - 2csc\phi uv + \cot\phi v^2], \quad (2)$$

and

$$\phi = \frac{a\pi}{2} \quad (3)$$

$$A_\phi = \sqrt{1 - \iota \cot\phi} \quad (4)$$

Unfortunately, the above defining integral can be rarely evaluated analytically [3; 5]. Both numerical integrations of quadratic exponentials and evaluation with spectral decomposition of the kernel could be expected to cost quadratic computational complexity  $\mathcal{O}(N^2)$ , where  $N$  is the time-bandwidth product of the input.

A preferred method, fast digital computation of fractional Fourier transform, was proposed in [8], without direct computation of Fresnel integrals. The samples of the transformed

function are obtained in terms of the samples of the original function, and the discrete form of FrFT is therefore given as

$$\mathcal{F}^a f\left(\frac{m}{2\Delta x}\right) = \frac{A_\phi}{2\Delta x} \sum_{n=-N}^N e^{i\pi(\alpha(\frac{m}{2\Delta x})^2 - 2\beta\frac{m+n}{(2\delta x)^2}) + \alpha(\frac{n}{2\Delta x})^2} f\left(\frac{n}{2\Delta x}\right) \quad (5)$$

To avoid the complexity  $\mathcal{O}(N^2)$ , in [8] Eq. (5) is converted into the summation of the convolution of  $e^{i\pi\beta(\frac{n}{2\pi})^2}$  and the chirp modulated function using some algebraic manipulations. With convolution computed with FFT, the overall complexity is  $\mathcal{O}(N \log N)$ . Furthermore, the authors showed that the samples of the continuous time fractional Fourier transform of a function can be approximately evaluated in terms of the samples of the original function in  $\mathcal{O}(N^2)$  time.

However, the oversampling of the continuous FrFT due to the ineffectiveness of Nyquist sampling criterion [5], will give a vector dimension so high that, we cannot afford complexity of  $\mathcal{O}(N \log N)$  in such situation. As for arbitrarily large discrete Fourier Transform, performing FFT is difficult for parallel processors for the reason of memory-bandwidth limitation, the challenge is even bigger in the case of FrFT [4]. Thus we propose to use tensor network to approximate the signal with fewer parameters.

Tensor networks enable a numerically reliable way to tackle the high-dimensional issue of the problem. As a special case of them, tensor-train (TT) [2] is able to represent high-dimensional tensor by a collection of smaller cores. Leveraging the promising expressive power of TT, we apply it to the proposed methods, in which TT is used to reduce the calculation with respect to Hadamard products. It is expected that the imposed TT format can further accelerate the digital computation of FrFT.

## 2 Methodology

In this work, we follow the study in [8] yet apply the tensor train (TT) [2] to further accelerate the computation of discrete fractional Fourier transform (frFT). Specifically, recall the formulas about ‘‘Method I’’ in [8]:

$$\mathbf{f}_a = (\mathbf{D}\mathbf{A}\mathbf{H}_{l_p}\mathbf{A}\mathbf{J}) \mathbf{f}, \quad (6)$$

where  $\mathbf{f}, \mathbf{f}_a \in \mathcal{C}^N$  denotes the input and output vector over the complex linear space, respectively. In Eq. (6),  $\mathbf{D}$  and  $\mathbf{J}$  are matrices representing the decimation and interpolation operations,  $\mathbf{\Lambda}$  denotes a diagonal matrix that corresponds to chirp multiplication, and  $\mathbf{H}_{l_o}$  corresponds to the convolutional operation.

Ignoring the decimation and interpolation operations for brevity, we find from Eq. (6) that the computation of FrFT consists of two aspects: (a) element-wise multiplication by given chirp signals; and (b) a convolution by  $\mathbf{H}_{l_o}$ . Note that in the vanilla method the complexity for the first aspect is equal to  $\mathcal{O}(N)$ , while the complexity for (b) equals  $\mathcal{O}(N \log N)$  because of FFT. However, the linear complexity  $\mathcal{O}(N)$  is also unacceptable when the size  $N$  is large. In this case, we need to take the acceleration for both (a) and (b) into account.

To accelerate the aspect (a), i.e., multiplication by a chirp signal, we reformulate the computation as a Hadamard product of two high-order tensors, which can be decomposed into low-rank tensor-train (TT) format. Assuming the vector  $\mathbf{v}$  containing the diagonal entries of  $\mathbf{\Lambda}$ . Denote by  $\underline{\mathbf{f}}$  and  $\underline{\mathbf{v}}$  the  $p$ th-order tensorized forms by rank- $r$  TT format. Thus, the Hadamard product of  $\underline{\mathbf{f}}$  and  $\underline{\mathbf{v}}$  can be obtained by directly computing the partial Kronecker product of their core tensors [6]. Specifically,

$$\underline{\mathbf{f}} \otimes \underline{\mathbf{v}} = \left( \underline{\mathbf{f}}^{(1)} \odot \underline{\mathbf{v}}^{(1)} \right) \times^1 \left( \underline{\mathbf{f}}^{(2)} \odot \underline{\mathbf{v}}^{(2)} \right) \times^1 \dots \times^1 \left( \underline{\mathbf{f}}^{(p)} \odot \underline{\mathbf{v}}^{(p)} \right), \quad (7)$$

where  $\otimes, \odot$  and  $\times^1$  denote the Hadamard product, partial Kronecker product and tensor contraction of two tensors, respectively.  $\underline{\mathbf{f}}^{(i)}$  and  $\underline{\mathbf{v}}^{(i)}$ ,  $i \in [p]$  denotes the core tensors of  $\underline{\mathbf{f}}$  and  $\underline{\mathbf{v}}$ , respectively. Note that the TT-rank for the above Hadamard product representation is equal to  $r^2$ . Hence the computation of Hadamard product in FrFT can be reduced when the rank  $r$  is small. As for the convolution aspect in Eq. (6), we apply the result in [12] to accelerate the required convolution operations, which is more efficient than the fast Fourier transform (FFT) method used in [8]. But it needs to be known that there exists approximation when using tensor networks including TT. Therefore, the corresponding approximation analysis is essential for this work in the future.

### 3 Go beyond digital approaches?

Another possible approach to overcome the limiting  $\mathcal{O}(N \log N)$  despite the high dimensionality is through optical implementation, departing from digital electronic system. Continual FrFT depicts the propagation of light. As light propagates, its distribution evolves through fractional transforms of increasing orders, passing through ordinary Fourier Transform [10; 11]. Optical realizations of discrete Fractional Fourier Transform can be realized by both classical and quantum optical systems presented in [14]. Digital computation of FrFT is used for comparison. Optical realization is attractive in its natural yield of  $\mathcal{O}(N)$ , however for now, the precision and accuracy are not yet there. Thus, tensor network is a practical approach in learning fractional Fourier Transforms and bettering wave simulations based on Fresnel diffraction theory in [1; 11].

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